Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

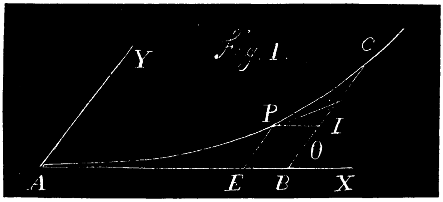
Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

CURVE OF PURSUIT GENERALIZED.

BY PROF. L. G. BARBOUR, RICHMOND, KY.

LET a particle start from B and describe a straight line BC with any given rate of motion, either uniform, or varying according to a given law. At the same instant let another particle start from A and maintain a velocity every where m times the velocity of the first particle, and a course always directed toward it; so that if the first particle should stop at any point as H, a tangent to the curve described by the other particle at the point P, which it has reached at the same instant, will pass through H.



Let AX be the axis of abscissas; A, the origin; AY parallel to BC, the axis of ordinates; AB, the unit of distance at the start; CBX, or θ, any angle from 0 to 180°.

The co-ordinates of P are $AE = x$ and $PE = y$. Draw PI parallel to AX. Let $AP = u$. By the conditions, $u = m.BH = m.BI + m.IH$. But

$$PI \;:\; IH \;::\; 1-x \;:\; IH \;::\; dx \;:\; dy;\ \therefore\ IH = (1-x)\frac{dy}{dx}.$$

It will be noticed that the obliqueness of the co-ordinates is recognized in the differentials. Therefore

$$u = m\left\{y + (1-x)\frac{dy}{dx}\right\};\ \therefore\ du = mdy + m(1-x)\frac{d^2y}{dx} - mdy = m(1-x)\frac{d^2y}{dx}$$

Also $du^2 = dx^2 + dy^2 + 2dxdy\cos\theta$. Therefore

$$m(1-x)\frac{d^2y}{dx} = dx\sqrt{\left(1 + \frac{dy^2}{dx^2} + 2\frac{dy}{dx}\cos\theta\right)}.$$

Let $v = dy \div dx$, then $m(1-x)dv = dx\,\sqrt{(1 + v^2 + 2v\cos\theta)}$. It must be remembered that θ is an arbitrary constant.

Separating the variables, $\dfrac{dv}{\sqrt{(1 + v^2 + 2v\cos\theta)}} = \dfrac{dx}{m(1-x)}$. Integrating,

$$\log[v + \cos\theta + \sqrt{(1 + v^2 + 2v\cos\theta)}] = -(1 \div m)\log(1-x) + C.$$

To determine the constant, let $x = 0$. This is at A. The curve at this pt. is tangent to axis of abscissas, $\therefore$ $(dy \div dx) = v = 0$; $\therefore$ $C = \log(\cos\theta + 1)$.

$$\therefore\ \log[v + \cos\theta + \sqrt{(1 + v^2 + 2v\cos\theta)}] = -(1 \div m)\log(1-x) + \log(\cos\theta + 1);$$

$$\therefore\ \frac{dy}{dx} + \cos\theta + \sqrt{(1 + v^2 + 2v\cos\theta)} = (1-x)^{-\frac{1}{m}}\left(\cos\theta + 1\right);$$

$$\therefore \quad \sqrt{\left(1+\frac{dy^2}{dx^2}+2\frac{dy}{dx}\cos\theta\right)} = (1-x)^{-\frac{1}{m}}\left(\cos\theta+1\right)-\frac{dy}{dx}-\cos\theta;$$

$$du = dx\sqrt{\left(1+\frac{dy^2}{dx^2}+2\frac{dy}{dx}\cos\theta\right)} = (1-x)^{-\frac{1}{m}}\left(\cos\theta+1\right)dx - dy - \cos\theta\, dx.$$

Integrating,

$$u = -\frac{(1-x)^{\frac{m-1}{m}}}{\frac{m-1}{m}}\left(\cos\theta + 1\right)-y-x\cos\theta+C'.$$

To find C', let $x=0$, $\therefore y=0$, $u=0$; $\therefore C'=[m\div(m-1)](\cos\theta+1)$.

Integrating between the limits $x=0$, $x=1$, and $y=0$, $y=BC$, we get

$$u = my = -y-\cos\theta+[m\div(m-1)](\cos\theta+1);$$

$$\therefore \quad y=\frac{m+\cos\theta}{m^2-1}; \ \therefore u = m\frac{m+\cos\theta}{m^2-1}.$$

Discussion. (1). If $\theta = 0$, $y = 1\div(m-1)$, $u = m\div(m-1)$. This is the well known algebraic problem of the couriers. BC becomes a prolongation of AB, and the curve APC degenerates into a straight line. It is readily seen that the hindmost particle travels over $AB = 1$, and in addition as far as the foremost particle moves, i. e.,

$$u=\frac{m}{m-1}=1+\frac{1}{m-1}.$$

If $\theta = 180°$, the two particles approach each other;

$$y=\frac{1}{m+1}; \ u=\frac{m}{m+1}; \ u+y=1.$$

If $\theta = 90°$, we have the ordinary form of the curve of pursuit;

$$y=\frac{m}{m^2-1}; \ u=\frac{m^2}{m^2-1}.$$

(2). If $m = 1$, $\theta = 180°$, then $y = u = \frac{1}{2}$. But if $m = 1$, $\theta = 0$, then $y = \infty$, as in the problem of the couriers.

If $m = 1$ and $\theta = 90°$, $y = \infty$, and the straight line BC is asymptote to the curve APC. The curve is then the *Tractrix*, whose evolute is the Catenary. When $m = \infty$, then $y = 0$, $u = 1 = AB$.

Let $m = 2$, $\theta = 0$, then $y = \frac{6}{6}$; $\theta = 60°$, $y = \frac{5}{6}$; $\theta = 90°$, $y = \frac{4}{6}$; $\theta = 120°$, $y = \frac{3}{6}$; $\theta = 180°$, $y = \frac{2}{6}$.

Ordinarily the particle starting from B is supposed to move at a constant rate. This, however has not entered into our demonstration at all. As stated in the outset, any rate may be assumed, provided both particles are subjected to the same law. We thus find easy solutions to problems like the following:

I. A falcon is trained to fly after a tennis ball that has been projected vertically, the bird heading always toward the ball.— Given the length of a straight line from the ball to the bird when he started, the angle that line makes with the vertical line described by the ball, and the velocity of the ball $= 1 \div m$ that of the bird.—Required the distance the falcon flies before catching the ball.

II. A hawk, carrying a fish in its mouth, is pursued by an eagle. At length both pause mid air for a moment. The hawk drops the fish, and the eagle, being on the alert, darts after it at the same instant. Same data and requirement as in No. I.

To find the value of u, x and y at any point of the curve, we can assume a value for x in the general equation which is

$$u = -\frac{m(1-x)^{\frac{m-1}{m}}}{m-1}(\cos\theta+1) - y - x\cos\theta + \frac{m}{m-1}(\cos\theta+1).$$

The value of y and of $u = my$ can be readily obtained.

But if we assume a value for y and then attempt to deduce that of x, it will make, in some cases, a difficult though not insoluble problem. It is best to remove the origin from A to B. Designate BE by x'; $\therefore 1 - x = x'$ and $-x = -1 + x'$. Substituting in the general equation, we have

$$u = my = -\frac{m}{m-1}x'^{\frac{m-1}{m}}(\cos\theta+1) - y - \cos\theta + x'\cos\theta + \frac{m}{m-1}(\cos\theta+1).$$

For any given values of m, θ and y, the value of x' can be found. The general form to which it can be reduced is

$$x'^{\frac{m-1}{m}} - ax = b.$$

Let $x' = z^m$; then $x'^{\frac{m-1}{m}} = z^{m-1}$; $\therefore z^{m-1} - az^m = b$; a, b and m being numbers.

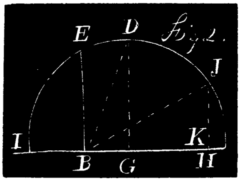
In the demonstration the origin might have been put at B; then dx would be a decreasing function of x, which is apt to perplex the signs.

Curve of Capture.—If a hare starts from B and runs toward A, and a hound at the same moment starts from A and runs toward B, they will meet at some intermediate point on the line AB. The particular point will be determined by the value of m in the expression $(m + \cos\theta) \div (m^2 - 1)$. Now let us make m the constant and θ the variable. For the purpose in hand, m must never be less than 1; if m be greater than 1, the hound will catch the hare, in whatever direction the hare may run from B, that is, whatever may be the value of θ. In fig. 1, θ is an acute angle, and the point of overtaking is at C. If θ were zero, the point would be on the axis of abscissas to the right of B. If in succession a hundred hares, running in a hundred different directions, should be pursued by as many hounds, the hares would be caught each at a different point; but the hundred points would all be situated on one and the same curve. It is of course understood that each hound runs m times as fast as the hare he pursues; m being an arbitrary constant, each value chosen would give a different curve but all the curves would belong to the same family.

In fig. 2, let B be the starting point of the hares. The locus of the p'ts where they are caught will be the *Curve of Capture.*

From the previous formula, generalizing and calling the distance from B to the point of capture ρ, we get the polar equation

$$\rho = \frac{m+\cos\theta}{m^2-1}. \quad \therefore\ IB = \frac{1}{m+1};\ BH = \frac{1}{m-1}.$$



Also $IB+BH = 2m\div(m^2-1)$. Make $IC = CH$ and we find

$$IC = \frac{m}{m^2-1};\ \therefore\ BC = \frac{1}{m^2-1}.$$

Since $BE = m\div(m^2-1)$, EBH being a right angle, $BE = IC$. Since $BC : IC :: 1 : m$, we may take any line as IH, bisect it at C, divide IC into m parts of which BC is one, and at B erect a perpendicular equal to IC; we shall have three points, I, E and H, of the curve. In this case, m is given. But we may suppose m not given. Then at any point of IC erect a perpendicular equal to IC, and we have three points of the Curve of Capture: Also $BC\div IC = m$.

We here suppose IC constant, and B a movable point. Place B at C, and we have $BC = 0$, $m = \infty$, and the curve becomes a circle. Remove C to an infinite distance, then $BC = IC$; $\therefore m = 1$, and the curve becomes a straight line.

To find the points where the curve cuts the axis of abscissas; any abscissa, as $BK = \rho\cos\theta = m\cos\theta + \cos^2\theta$, $\therefore -m\sin\theta - 2\cos\theta\sin\theta = 0$, $\therefore \sin\theta = 0$, $\therefore \theta = 0$ or $180°$, for the maximum value of abscissas. The curve is perpendicular to the axis at I and H. Or the equation may be satisfied by $-m-2\cos\theta = 0$; $\therefore \cos\theta = -\frac{1}{2}m$, which gives negative values for cosine, and $\theta > 90°$; our limits forbid the consideration of these.

The maximum and minimum values of ρ are easily derived from

$$\rho = \frac{m+\cos\theta}{m^2-1};$$

$\therefore -\sin\theta = 0$; $\therefore \theta = 0$, or $180°$, as we found in the case of abscissas.

From the genesis of the curve it is evidently symmetrical to the axis of abscissas. The figure is drawn as if CD were the maximum ordinate. Let us see if this is true. $BC = 1\div(m^2-1) = BD\cos\theta$; therefore

$$\frac{m\cos\theta+\cos^2\theta}{m^2-1}=\frac{1}{m^2-1};\ \therefore\ \cos\theta=\frac{-m\pm\surd(m^2+4)}{2}.$$

This is the value of $\cos\theta$ when $\theta=DBC$. To see whether CD is a maximum ordinate, take any ordinate, as $JK=JB\sin\theta=\rho\sin\theta=(m\sin\theta+\cos\theta\sin\theta)\div(m^2-1)$; for a maximum, $m\cos\theta+\cos^2\theta=\sin^2\theta=1-\cos^2\theta$. $\therefore 2\cos^2\theta-\frac{1}{2}m\cos\theta=\frac{1}{2}$; $\therefore \cos\theta=\frac{1}{4}[-m\pm\surd(m^2+8)]$, which differs from the value for $\theta=DBC$. It will be found moreover that no value for m will satisfy both equations. Therefore the curve is not symmetrical to the axis CD. Thus for $m=2$, we get $DBC=65°32'$; whereas, for the max. ordinate, $\theta=68°31'$; showing that the highest point is to the left of D.

SOLUTION OF A PROBLEM IN MENSURATION, BY REQUEST. *Problem.*—Four cylinders of equal radii, r, intersect each other at right angles, two and two, in the same horizontal plane, each pair being tangent to each other and having their axes parallel. Required the contents of the solid bounded by the surfaces of the four cylinders and their common tangent plane.

Solution.—Let $ADEF$ represent one-fourth of the surface bounded by the common tangent plane, and $ADEFB$, one fourth of the required solid; then will ABC and BEG represent one-fourth the sectional areas of two of the intersecting cylinders.

Let $adef$ be a section of the required solid parallel to the common tangent plane and cutting the radius CA in H, and put $AH=x$. Then from property of the circle, we have

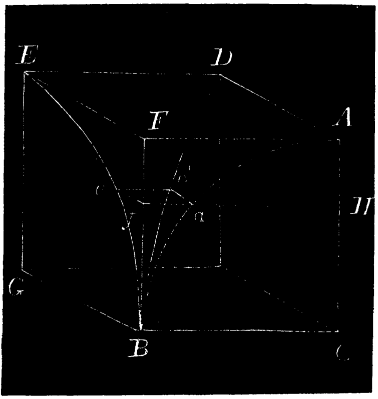
$$aH=\surd(2rx-x^2),$$

and consequently

$$fa=r-\surd(2rx-x^2),$$

= one of the equal sides of the variable plane $adef$. Hence the area of the variable plane $adef=[r-\surd(2rx-r^2)]^2$

$$=r^2+2rx-x^2-2r\surd(2rx-x^2).$$



Because the required solid is the *sum* of all its parallel planes, putting dx for the thickness of each plane and S for one-fourth the required solid, we have

$$dS=r^2dx+2rxdx-x^2dx-2r\surd(2rx-x^2)dx;$$

$$\therefore S=r^2x+rx^2-\tfrac{1}{3}x^3-\int_0^r\surd(2rx-x^2)dx$$

$$=\tfrac{5}{3}r^3-\tfrac{1}{2}r\times\text{area of circle whose radius is } r.$$

$$\therefore 4S=\tfrac{20}{3}r^3-2r\times\text{area of circle whose radius is } r.$$

If $r=$ 7ft 3in, this becomes $2540.52-2394.39=146.13$ cnbic feet.